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The Convergence and Continuity of Rational Functions Closely Related to Padé Approximants

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Given any meromorphic function f, let E be a compact subset of C not containing any poles of f. It is shown that sequences of rational functions obtained by deleting certain poles of diagonal sequences of Pade approximants of f converge uniformly to f on E. Also discussed are existence and continuity properties of this pole deletion approximation procedure.

1. INTRODUCTION

Padé approximants have been widely used in numerical analysis, physics and engineering. However, there is a variety of unanswered questions concerning the convergence of sequences of Padé approximatnts. For example, it is not known, for a meromorphic function f, if some subsequence of a diagonal sequence of Padé approximants converges uniformly to f. We contribute to the study of Padé approximants by showing for any meromorphic function f that sequences of rational functions closely related to diagonal sequences of Padé approximants converge uniformly to f on compact sets containing no poles of f. We also discuss continuity properties and existence properties of our rational functions.

To describe our rational functions we will use the Frobenius definition of Padé approximants. Given a formal power series $f(z) = \sum_{i=0}^{\infty} d_i z^i$ the (n/m) Padé approximant is defined as the unique rational function for which

$$P_{mn}(z) = \frac{p_{mn}(z)}{q_{mn}(z)}$$

where

$$p_{mn}(z) = q_{mn}(z) f(z) + O(z^{m+n+1}) \qquad (z \to 0) \tag{1}$$

and where $p_{mn}(z)$ and $q_{mn}(z)$ are polynomials, respectively, of degrees $\leqslant n$ and $\leqslant m$. For later convenience given $P_{mn}(z)$ we define $p_{mn}^*(z)$ and $q_{mn}^*(z)$

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to be the unique polynomials with no common factors such that $q_{mn}^*(0) = 1$ and

$$P_{mn}(z) = \frac{p_{mn}^*(z)}{q_{mn}^*(z)}.$$

This $q_{mn}^*(z)$ will be called the minimal denominator for $P_{mn}(z)$. It is known [4] that for any $m \geqslant 0$ and $n \geqslant 0$ and any $f(z)$ with a power series expansion that $P_{mn}(z)$, $p_{mn}^*(z)$ and $q_{mn}^*(z)$ exist and are unique. To define our approximant assume that

$$P_{mn}(z) = \frac{p_{mn}^*(z)}{q_{mn}^*(z)}$$

is known and it has a partial fraction decomposition

$$P_{mn}(z) = r_{mn}^0(z) + \sum_{i=1}^{M} \frac{r_{mn}^i(z)}{(z - \zeta_i)^{k_i}}$$

where ζ_i, $i = 1, 2, \ldots, M$ are the zeroes each of multiplicity k_i of $q_{mn}^*(z)$, $r_{mn}^0(z)$ is a polynomial and $r_{mn}^i(z)$, $i = 1, 2, \ldots, M$, are polynomials, respectively, of degree less than k_i. Let $S_{m,n}$ be any set in $\mathbb{C}$. Then our (n/m) rational approximant is defined to be

$$R_{mn}(z) = \Gamma_{S_{m,n}}[P_{mn}(z)] = r_{mn}^0(z) + \sum_{\zeta_i \notin S_{m,n}} \frac{r_{mn}^i(z)}{(z - \zeta_i)^{k_i}}. \tag{2}$$

We note that since Padé approximants as defined by Frobenius always exist then it follows easily that for any $f(z)$ with a power series expansion, any $m \geqslant 0$, any $n \geqslant 0$ and any set $S_{m,n}$ our (n/m) approximant is well defined.

Later we will discuss literature relevant to our approach. However, here we note that Walsh in [6] discuss a pole elimination scheme similar to (2). His results, however, are directed towards rational functions other than Padé approximants and apply only to rational functions of fixed denominator or fixed numerator degree.

2. Continuity and Convergence

In this section we study the continuity of our approximants as a function of co-efficients in the power series expansion of a given function and study the pointwise convergence of our approximants to a meromorphic function f.

To do this we first present a useful lemna.

Lemma 1. *Let $L \geqslant 0$ and points ζ_i, $i = 1, 2, \ldots, L$,* in *$\mathbb{C}$ be given. Let Δ and R be chosen so that $R \geqslant 1$, $\Delta \leqslant 1$ and circles of radius 3Δ about each ζ_i*

are contained in the circle $|z| \leqslant R$. *Then for any* $m \geqslant 0$ *and any real numbers* b_i, $i = 0, 1, 2,..., m$ *there exists a curve* C_m *(depending on the* b_i*'s and* m*) consisting of* L *circles* $C_{i,m}$, *centered, respectively, at* ζ_i, $i = 1, 2,..., L$, *and of radius between* Δ *and* 3Δ *and a circle* $C_{0,m}$ *with center at* $z = 0$ *and radius between* $R - 2\Delta$ *and* R *such that*

$$\max_{z \in C_m} \frac{\sum_{j=0}^{m} |b_j z^j|}{|\sum_{j=0}^{m} b_j z^j|} \leqslant \left(\frac{24eR}{\Delta}\right)^m.$$

Proof. The proof is not difficult using Cartan's lemma [3] and some of the techniques used in [1, 5, 7, 8, 9].

Since $24eR/\Delta > 1$ we can assume without loss of generality that $b_m \neq 0$ and therefore that $b_m = 1$. We let $p(z) = \sum_{j=0}^{m} b_j z^j = \prod_{j=1}^{m} (z - z_j)$ and assume

$$z_j \leqslant 2R, j = 1, 2,..., \ell, \text{ and} \tag{3}$$

$$z_j > 2R \text{ for } j = \ell + 1,..., m. \tag{4}$$

Consider

$$\frac{|b_k z^k|}{|\sum_{j=0}^{m} b_j z^j|} = \frac{|z|^k \, |\sum_{\Omega_{m-k}} (\prod_{j \in \Omega_{m-k}} z_j)|}{|\prod_{j=1}^{\ell} (z - z_j) \prod_{j=\ell+1}^{m} (z - z_j)|} \tag{5}$$

where Ω_{m-k} indicates any set of $m - k$ distinct elements of $\{1, 2,..., m\}$ and the sum is over all Ω_{m-k} (m and k fixed). Since for any k there are $\binom{m}{k}$ such Ω_{m-k}, since $R > 1$ and by (3), (4) and (5) it follows for $|z| < R$ that

$$\frac{|b_k z^k|}{|\sum_{j=0}^{m} b_j z^j|} \leqslant \binom{m}{k} \frac{R^k (2R)^{m-k}}{|\prod_{j=1}^{\ell} (z - z_j)|} \prod_{j=\ell+1}^{m} \left|\frac{z_j}{z - z_j}\right|. \tag{6}$$

Since $|z| \leqslant R$ and $|z_j| \geqslant 2R, j = \ell + 1,..., m$ and by (6) it follows that

$$\frac{\sum_{j=0}^{m} |b_j z^j|}{|\sum_{j=0}^{m} b_j z^j|} \leqslant \frac{(3R)^m \, 2^{m-\ell}}{|\prod_{j=1}^{\ell} (z - z_j)|} \leqslant \frac{(6R)^m}{|\prod_{j=1}^{\ell} (z - z_j)|}.$$

We now apply Cartan's Lemma [3] to conclude that except inside of ℓ circles the sum of whose diameters is at most Δ,

$$\left|\prod_{j=1}^{\ell} (z - z_j)\right| \geqslant \left(\frac{\Delta}{4e}\right)^{\ell} \geqslant \left(\frac{\Delta}{4e}\right)^m.$$

Since the sum of the diameters of these circles is less than Δ it follows that it is always possible to choose the curve C_m of the lemma so that for $z \in C_m$

$$\frac{\sum_{j=0}^{m} |b_j z^j|}{|\sum_{j=0}^{m} b_j z^j|} \leqslant \frac{(6R)^m}{\left(\frac{\Delta}{4e}\right)^m} \leqslant \left(\frac{24eR}{\Delta}\right)^m. \quad \blacksquare$$

We now consider the continuity of our approximants as a function of the coefficients in a power series expansion:

THEOREM 1. *Let $f_k(z) = \sum_{i=0}^{\infty} d_{ki} z^i$, $i = 0, 1, 2,\ldots$, be a formal power series, let E be any compact set in $\mathbb{C}$, let $m \geqslant 0$ and $n \geqslant 0$ and let $P_{mnk}(z)$ be the (n/m) Pade approximant to $f_k(z)$. If $d_{ki} \to d_{0i}$, $i = 0, 1,\ldots, m + n$, as $k \to \infty$ then there exists sets S_k in $\mathbb{C}$ such that*

$$\Gamma_{S_k}[P_{mnk}(z)] \to \Gamma_E[P_{mn0}(z)]$$

uniformly on E.

Proof. First we describe the sets S_k of the theorem. Let $p_{mnk}(z)$ and $q_{mnk}(z)$ be any polynomials, respectively, of degrees $\leqslant n$ and $\leqslant m$ which satisfy (1) with f replaced by f_k. Also let $p_{mnk}^*(z) = \sum_{i=0}^{n} a_{ki} z^i$ and $q_{mnk}^*(z) = \sum_{i=0}^{m} b_{ki} z^i$, $q_{mnk}(0) = 1$ be the polynomial obtained when common factors are omitted from $p_{mnk}(z)$ and $q_{mnk}(z)$. Since E is compact we may choose a disc $|z| \leqslant R$, $R \geqslant 1$, such that E is interior to this disc and such that $|z| = R$ contains no zroes of $q_{mn0}^*(z)$. Let ζ_i, $i = 1, 2,\ldots, \alpha (\alpha \leqslant m)$ be the zeroes of $q_{mn0}^*(z)$ and let ζ_i, $i = 1, 2,\ldots, L$, be the zeroes of $q_{mn0}^*(z)$ not in E and such that $|\zeta_i| < R$. Now we choose Δ so that

$\Delta \leqslant 1$; the discs $|z - \zeta_i| \leqslant 4\Delta$, $i = 1, 2,\ldots, L$, and the annulus $R - 3\Delta \leqslant |z| \leqslant R$ *are disjoint and do not intersect E; and such that* $R \leqslant |z| \leqslant R + \Delta$ contains none of ζ_i, $i = 1, 2,\ldots, \alpha$. (7)

We now apply Lemma 1 to construct curves C_m^k, $k = 0, 1,\ldots$, such that

$$\max_{z \in C_m^k} \frac{\sum_{j=0}^{m} |b_{kj} z^j|}{|\sum_{j=0}^{m} b_{kj} z^j|} \leqslant \left(\frac{24eR}{\Delta}\right)^m. \tag{8}$$

Note that C_m^k consists of circles $C_{0,m}^k$, $C_{1,m}^k$,..., $C_{L,m}^k$ as in the lemma. We define S_k to be the region interior to $C_{0,m}^k$ and exterior to $C_{i,m}^k$, $i = 1, 2,\ldots, L$.

First, note that by our construction S_k and E contain precisely the same set of zeroes of $q_{mn0}^*(z)$ so that

$$\Gamma_{S_k}[P_{mn0}(z)] = \Gamma_E[P_{mn0}(z)]. \tag{9}$$

Next we let w_k be the highest power of z common to $p_{mnk}(z)$ and $q_{mnk}(z)$. Note by (1) that

$$p_{mnk}^*(z) = q_{mnk}^*(z) f_k(z) + O(z^{m+n+1-w_k}), \qquad k = 0, 1, 2,\ldots \tag{10}$$

and that $p_{mnk}^*(z)$ and $q_{mnk}^*(z)$, respectively, are of degrees $\leqslant n - w_k$ and $\leqslant m - w_k$.

Now

$$P_{mnk}(z) - P_{mn0}(z) = \frac{p^*_{mnk}(z)\, q^*_{mn0}(z) - p^*_{mn0}(z)\, q^*_{mnk}(z)}{q^*_{mnk}(z)\, q_{mn0}(z)} \tag{11}$$

and by (10)

$$= \frac{q^*_{mnk}(z)\, q^*_{mn0}(z)[\sum_{i=0}^{\infty} (d_{ki} - d_{0i})\, z^i] + O(z^{m+n+1-w_k}) + O(z^{m+n+1-w_0})}{q^*_{mnk}(z)\, q^*_{mn0}(z)}. \tag{12}$$

But the numerator of (11) is a polynomial of degree less than $m + n - w_0 - w_k$ and therefore (12) implies

$$|P_{mnk}(z) - P_{mn0}(z)|$$

$$\leqslant \frac{\sum_{i=0}^{m} |b_{ki}z^i| \sum_{i=0}^{m} |b_{0i}z^i| (\sum_{i=0}^{m+n} |d_{ki} - d_{0i}|\,|z|^i)}{|\sum_{i=0}^{m} b_{ki}z^i|\,|\sum_{i=0}^{m} b_{0i}z^i|}. \tag{13}$$

Note that (13) and its derivation is largely the same as that of (3.9) of [5]. However equation (13) is more general in that we have used the Frobenius definition of Pade approximant.

The key concept in our proof involves the Cauchy integral formula. In particular, note that S_k is bounded by C_m^k — a finite number of simple Jordan curves none of which lie on any poles of $P_{mnk}(z)$. It follows from Cauchy's integral formula that for z interior to S_k

$$\Gamma_{S_k}[P_{mnk}(z)] = \frac{1}{2\pi i} \int_{C_m^k} \frac{P_{mn0}(\sigma)}{z - \sigma}\, d\sigma$$

Also C_m^k lies on no poles of $P_{mn0}(z)$ so

$$\Gamma_{S_k}[P_{mn0}(z)] = \frac{1}{2\pi i} \int_{C_m^k} \frac{P_{mn0}(\sigma)}{z - \sigma}\, d\sigma$$

By these equations and (9);

$$|\Gamma_{S_k}[P_{mnk}(z)] - \Gamma_E[P_{mn0}(z)]|$$

$$= \left| \frac{1}{2\pi i} \int_{C_m^k} \frac{P_{mnk}(\sigma) - P_{mn0}(\sigma)}{z - \sigma}\, d\sigma \right|$$

$$\leqslant \frac{1}{2\pi} \int_{C_m^k} \left| \frac{P_{mnk}(\sigma) - P_{mn0}(\sigma)}{z - \sigma} \right| |d\sigma|$$

and by (13) and the definitions of Δ and $C_m{}^k$

$$\leqslant \frac{2\pi L3\Delta + 2\pi R}{2\pi\Delta} \max_{z\in C_m{}^k} | P_{mnk}(z) - P_{mn0}(z)|$$

$$\leqslant \left(3L + \frac{R}{\Delta}\right) \max_{z\in C_m{}^k} \frac{\sum_{i=0}^m | b_{ki}z^i | \, | \sum_{i=0}^m b_{0i}z^i | \sum_{i=0}^{m+n} | d_{ki} - d_{0i} | \, | z^i |}{| \sum_{i=0}^m b_{ki}z^i | \, | \sum_{i=0}^m b_{0i}z^i |}.$$

Finally since the union of $C_m{}^k$, $k = 0, 1, 2,\ldots$, is contained in a compact set bounded (by distance Δ) away from zeroes of $q_{mn0}^*(z)$ and interior to or on $| z | \leqslant R$ and applying Lemma 1, we obtain for z in E

$$| \Gamma_{S_k}[P_{mnk}(z)] - \Gamma_E[P_{mn0}(z)]|$$

$$\leqslant \left(3L + \frac{R}{\Delta}\right)\left(\frac{24eR}{\Delta}\right)^m BR^{n+m} \left(\sum_{i=0}^{m+n} | d_{ki} - d_{0i} |\right) \tag{14}$$

where B does not depend on k. Since m, n, R, L and Δ are also independent of k, the theorem follows. ∎

COROLLARY 1. *In addition to the assumptions of Theorem* 1 *assume that* E *contains no poles of* $P_{mn0}(z)$. *Then*

$$\Gamma_{S_k}[P_{mnk}(z)] \to P_{mn0}(z)$$

uniformly on E.

Proof. From the definition of $\mathscr{C}$ and our assumption on E

$$P_{mn0}(z) = \Gamma_E[P_{mn0}(z)]. \quad ∎$$

We note, as will be illustrated later, that there exist sequences of Padé approximants $P_{mnk}(z)$, $k = 0, 1, 2,\ldots$, and sets E which satisfy the assumptions of Theorem 1 and Corollary 1 but for which

$$P_{mnk}(z) \not\to P_{mn0}(z) \text{ on } E.$$

Although we do not present a proof, we remark that to insure such Padé approximants $P_{mnk}(z)$ approach $P_{mn0}(z)$ on E one needs an additional condition that a determinant involving d_{0i}, $i = 0, 1,\ldots, m + n$, is non-zero. Corollary 1 and Theorem 1 are important because no restrictions on $m \geqslant 0$, $n \geqslant 0$ and d_{0i}, $i = 0, 1,\ldots, m + n$ are required. Corollary 1 and Theorem 1 later will be illustrated.

We now consider the convergence of our approximants.

THEOREM 2. *Let $f(z)$ be analytic at $z = 0$ and meromorphic in $\mathbb{C}$, let $\lambda > 0$ and let $P_{m_\nu,n_\nu}(z)$ be any sequence of Padé approximants of $f(z)$ with*

$$n_\nu \geqslant \lambda m_\nu . \tag{15}$$

Let E be a compact set containing no poles of $f(z)$. There exists sets S_{m_ν,n_ν} in $\mathbb{C}$ such that

$$\Gamma_{S_{m_\nu,n_\nu}}[P_{m_\nu,n_\nu}(z)] \to f(z)$$

uniformly on E as $m_\nu \to \infty$.

Proof. The proof will follow from our equation (14) and certain results contained in [5] and [7]. For completeness as indicated we will repeat some portions of the proofs in [5] and [7]. For simplicity in our proof we will drop the subscripts ν when referring to $P_{m_\nu,n_\nu}(z)$, S_{m_ν,n_ν}, etc. However, by the sequence $P_{m,n}(z)$ or the sets $S_{m,n}$, etc., we mean that the pairs m, n form a sequence m_ν, n_ν, $\nu = 1, 2,\ldots$, as in the theorem statement.

We first define the sets $S_{m,n}$ and other parameters which we need later. We let ζ_i, $i = 1, 2,\ldots, \alpha$, be the poles of $f(z)$ where $\alpha \geqslant 0$ and if $f(z)$ has an infinite number of poles, $\alpha = \infty$. It is assumed that if a pole has order > 1 then it is repeated in the sequence ζ_i, $i = 1, 2,\ldots, \alpha$. We also assume $|\zeta_1| \leqslant |\zeta_2| \leqslant |\zeta_3|$, etc., and we choose $R \geqslant 1$ so that $|z| < R$ contains E and $|z| = R$ does not contain any ζ_i, $i = 1, 2,\ldots, \alpha$. Let L be the number of ζ_i's in $|z| < R$ and choose Δ satisfying (7). We select the sets $S_{m,n}$ exactly as S_k are defined in the proof of Theorem 1 except the polynomials $\sum_{j=0}^{m} b_{kj}z^j$ in the definition of S_k should be replaced by minimal denominators of the Padé approximants $P_{mn}(z)$. Finally, we choose ρ large enough so that $R^2/\rho \leqslant 1$ and

$$\left(\frac{24eR^2}{\Delta}\right)\left(1 + \frac{R}{|\zeta_1|}\right)\left(\frac{R^2}{\rho}\right)^\lambda \leqslant \frac{1}{2} \tag{16}$$

and we let M be the number of ζ_i satisfying $|\zeta_i| \leqslant \rho$.

By the definition of ρ and M it follows that we can express $f(z)$ as

$$f(z) = \frac{\sum_{i=0}^{\infty} c_i z^i}{\prod_{i=1}^{M}\left(1 - \frac{z}{\zeta_i}\right)} \tag{17}$$

where $\sum_{i=0}^{\infty} c_i z^i$ is uniformly convergent for $|z| \leqslant \rho$. We also define for $n \geqslant 0$

$$f_n(z) = \frac{\sum_{i=0}^{n} c_i z^i}{\prod_{i=1}^{M}\left(1 - \frac{z}{\zeta_i}\right)} \tag{18}$$

and for $m \geqslant 0$ we let $P_{mn0}(z)$ be the Padé approximant to $f_n(z)$. It follows easily that for $m \geqslant M$, $P_{mn0}(z) = f_n(z)$ and thus for $m \geqslant M$ that the minimal denominator of $P_{mn0}(z)$ will be the fixed polynomial $q^*_{mn0}(z) = \prod_{i=1}^{m}(1 - z/\zeta_i)$. Finally we define d_i, $i = 0, 1,\ldots$, and d_{ni}, $i = 0, 1,\ldots$, by

$$f(z) = \sum_{i=0}^{\infty} d_i z^i, \tag{19}$$

$$f_n(z) = \sum_{i=0}^{\infty} d_{ni} z^i. \tag{20}$$

Precisely as we derived (14) it follows that for $m \geqslant M$ and $z \in E$

$$\begin{aligned} &|\Gamma_{S_{m,n}}[P_{mn}(z)] - P_{mn0}(z)| \\ &\quad = |\Gamma_{S_{m,n}}[P_{mn}(z)] - \Gamma_E[P_{mn0}(z)]| \\ &\quad \leqslant \left(3L + \frac{R}{\Delta}\right)\left(\frac{24eR}{\Delta}\right)^m BR^{m+n}\left(\sum_{i=0}^{m+n} |d_i - d_{ni}|\right). \end{aligned} \tag{21}$$

Since for $m \geqslant M$ $q^*_{mn0}(z)$ is a fixed polynomial, it follows that B will not depend on m or n.

Now following [5] we note that by the uniform convergence of $\sum_{i=0}^{\infty} c_i z^i$ there exists a constant L such that

$$c_i \leqslant L\rho^{-i}, \; i = 0, 1, 2,\ldots, \tag{22}$$

Also we define

$$\left[\prod_{i=1}^{M}\left(1 - \frac{z}{\zeta_i}\right)\right]^{-1} = \sum_{i=0}^{\infty} g_i z^i. \tag{23}$$

Then from (17-20), (22) and (23) it is not difficult to show that for $m \geqslant M$

$$\sum_{i=0}^{n+m} |(d_i - d_{ni}) z^i| \leqslant L \left|\frac{z}{\rho}\right|^{n+1} m \sum_{i=0}^{m-1} |z|^i |g_i| \tag{24}$$

(see 3.4 of [5]). As in [5], by the definition of ζ_i and by (23) it follows that each term in $\sum_{i=1}^{m-1} |z|^i |g_i|$ will be dominated by terms in the expansion of $(1 - |z/\zeta_1|)^{-M}$. From this fact and by (24) it then follows that for $m \geqslant M$

$$\sum_{i=0}^{n+m} |(d_i - d_{ni}) z^i| \leqslant L \left|\frac{z}{\rho}\right|^{n+1} m \frac{(M + m - 2)^{M-1}}{(M - 1)!} \sum_{i=0}^{m-1} \left|\frac{z}{\zeta_1}\right|^i$$

(see 3.5 of [5]). From this equation and since $z \in E$ implies $|z| \leq R$ it follows easily that

$$\left|\sum_{i=0}^{n+m} (d_i - d_{ni}) z^i\right| \leq Am^M \left(\frac{R}{\rho}\right)^n \left(1 + \frac{|z|}{|\zeta_1|}\right)^m \tag{25}$$

(as in [7]) where A does not depend on m and n.

Now by (16), (21) and (25) it can be concluded that for $m \geq M$ and $z \in E$

$$|\Gamma_{S_{m,n}}[P_{mn}(z)] - f(z)| \leq \left(3L + \frac{R}{\Delta}\right) m^M AB \left(\frac{1}{2}\right)^m + |f_n(z) - f(z)|. \tag{26}$$

Finally, it follows easily from the definitions of $f(z)$ and $f_n(z)$ that for $z \in E$, $|f_n(z) - f(z)| \to 0$ as $n \to \infty$. Since $n \geq \lambda m$ the theorem follows from (26). ∎

We remark that although Theorem 1, Corollary 1 and Theorem 2 require selection of an appropriate set, S_k or S_{m_ν, n_ν} this selection appears not to be difficult. In fact in most practical problems the choice of S_k or S_{m_ν, n_ν} equal to E will suffice. In any case there are only a finite number of poles to a Padé approximant and in the examples we have considered it was easy to select which of these to eliminate. If required, the proof of Cartan's Lemma [3] can be used to constructively select S_k or S_{m_ν, n_ν}. We also remark that application of $\Gamma_{S_{m,n}}$ as defined in (2) to a rational function requires a partial fraction decomposition of the rational function in addition to construction of the Padé approximant. Usually this is not a difficult computational problem.

3. Examples and Discussion

Example 1. To see the importance of continuity results such as Theorem 1 and Corollary 1 consider the functions

$$f_k(z) = \frac{2}{2-z} + \frac{1}{k} \frac{121 + 86z + 243z^2 + 472z^3}{(2-z)(3-z)(4-z)(5-z)}, \qquad k \geq 1,$$

and $P_0(z) = 2/(2-z)$ and the set $E = \{z : |z| \leq 1\}$. Suppose, for the sup norm over E and $1/k$ "sufficiently" small, we wish to approximate $f_k(z)$ by a rational function of denominator degree not greater than 2 and numerator degree not greater than 1. If we consider the (1/2) Padé approximant $P_k(z)$ to $f_k(z)$ we obtain

$$P_k(z) = \frac{2 + \frac{1}{k}}{2-z} + \frac{1}{k} \frac{1}{1-2z}$$

which is a poor approximation to $f_k(z)$ in the sup norm because of the pole at

$z = \frac{1}{2}$. As can be seen (see the next paragraph) this difficulty arises from the fact that Padé approximation does not satisfy Theorem 1 or Corollary 1.

However, let us now consider our (1/2) approximant $R_k(z)$ as described in Theorem 1 to $f_k(z)$. For $1/k$ sufficiently small it follows easily that the first four power series coefficients of $f_k(z)$ will be close to those of $P_0(z)$. Therefore, it follows by Corollary 1 that for $1/k$ sufficiently small $R_k(z)$ will closely approximate $P_0(z)$ over E. However, it is clear for $1/k$ sufficiently small that $P_0(z)$ is close to $f_k(z)$ throughout E. Thus it follows that for $1/k$ sufficiently small $R_k(z)$ will be close to $f_k(z)$ throughout E. For example, selecting the set S_k of Theorem 1 equal to E

$$R_k(z) = \frac{2 + \dfrac{1}{k}}{2 - z}$$

and for $k = 1000$ (say) we can show that

$$| R_k(z) - f_k(z) | \leq .038.$$

Thus for $1/k$ sufficiently small our approximant to $f_k(z)$ is a good approximation although the (1/2) Padé approximant is a poor approximation to $f_k(z)$. ∎

EXAMPLE 2. This example illustrates Theorem 2 using a version of Gammel's [2] example. In particular following Gammel we define

$$f(z) = \sum_{j=0}^{\infty} f_j z^j = 1 + \sum_{\nu=1}^{\infty} \alpha_\nu \sum_{j=n_\nu}^{2n_\nu} r_\nu^j z^j$$

where $n_{\nu+1} = 2n_\nu + 1$, $n_0 = 0$,

$$\alpha_\nu = \frac{1}{(2n_\nu)!} \, | r_\nu |^{2n_\nu},$$

and $r_\nu \neq 0$ is any sequence in $| z | \leq 1$ whose limit points are dense (say) in $| z | \leq 1$. Again we choose $E = \{z : | z | \leq 1\}$ and we let $P_{n_\nu n_\nu}(z)$ be the (n_ν/n_ν) Padé approximant to $f(z)$ and $R_{n_\nu n_\nu} = \Gamma_E[P_{n_\nu n_\nu}(z)]$. Then it is known [2] that

$$P_{n_\nu n_\nu}(z) = \sum_{j=0}^{n_\nu - 1} f_j z^j + \alpha_\nu z^{n_\nu} r_\nu^{-n_\nu} \left(1 - \frac{z}{r_\nu}\right)^{-1}$$

converges pointwise to $f(z)$ nowhere in E. However it is easily shown that

$$R_{n_\nu n_\nu} = \sum_{j=0}^{n_\nu - 1} f_j z^j - \alpha_\nu r_\nu^{-n_\nu + 1} \sum_{j=0}^{n_\nu - 1} r_\nu^{n_\nu - 1 - j} z^j$$

and then it may be directly verified that $R_{n_\nu n_\nu} \to f(z)$ uniformly in E. In fact noting that $f(z)$ is entire we can conclude by applying Theorem 2 that for appropriate sets $S_{n,n}$ the entire sequence $\Gamma_{S_{n,n}}[P_{nn}(z)]$ will converge to $f(z)$ uniformly in E. ∎

Finally we wish to make a few further remarks and briefly outline the related literature. First we remark that Theorem 2 can be improved to allow $f(z)$ to have a finite number of essential singularities in compact subsets of $\mathbb{C}$. In fact for certain subsequences of our approximants such a result follows rather directly from results appearing in [1]. However, since we wished to consider any sequences of our approximants satisfying (15) and to achieve conciseness we have chosen the development of Section 2. As another extension of our results we note without presenting any details that a pole elimination scheme such as ours can be usefully applied to other rational approximation procedures such as Newton-Padé or Cauchy approximation [11] (see [10]).

In addition to the article of Walsh discussed earlier there are a variety of other articles related to our results. Relevant to our results on continuity, a number of papers discuss continuous dependence in rational approximation procedures. A variety of papers [13]-[15] describe continuity properties for best Chebyshev rational approximation and [11] and [13] prove continuity results similar to Theorem 1 for Newton–Padé approximation. However, all these papers require some type of "normality" condition—such as requiring a determinant be non-zero—for their pointwise continuity results. As mentioned earlier our results do not require normality conditions. Also [16], [17] and [18] contain results related to our Theorem 1. In fact, in [16] Chui, Shisha, and Smith avoid requiring any normality conditions, but the results of [16], [17] and [18] are all directed toward the convergence of certain best rational approximants to Padé approximants and thus, although related, are different from our results. An interesting possible extension of Theorem 1 would be to consider the effect of pole elimination on best Chebyshev rational approximation.

Relevant to our results on pointwise convergence of Padé approximants Chisholm [5] and Beardon [7] have presented results but their results require technical assumptions about the location of poles of the Padé approximants. These assumptions are known not necessarily to be satisfied for the entire sequence of (n/n) Padé approximants to a meromorphic $f(z)$ and it is unknown, as yet, if they are satisfied for subsequences of (n/n) Padé approximents. In fact, this is Baker's [12, 1] unproven conjecture. Some other relevant results are those of Nuttal [9] and Pommerenke [8]. However these results concern convergence in measure and capacity, not pointwise convergence. Also we note that in [5] Chisholm shows that for meromorphic $f(z)$, for each pole of $P_{mn}(z)$ (m, n large) which is not "near" a pole of $f(z)$

there will be a "nearby" zero to $P_{mn}(z)$. Although Chisholm's result is closely related to our results he does not discuss pole elimination. Baker's book [1] discusses much of the recent literature.

Finally, we re-emphasize the importance of our pole elimination scheme—it allows construction of rational approximants closely related to Padé approximants, that have desirable uniform pointwise convergence properties and desirable continuity properties.

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